HOMOMORPHISMS OF TRANSFORMATION GROUPS[1]

BY

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Introduction. Let (X, T) be a transformation group with compact phase space X and with arbitrary phase group T. We first point out in Theorem 1 that there exist least invariant closed equivalence relations S_d and S_e in X such that T is distal on $X|S_d$ and T is equicontinuous on $X|S_e$. This permits, so to speak, the dividing out of certain more complicated parts of a transformation group. The application of this process to properties other than distal and equicontinuous is indicated by Remark 8. Theorem 2 then relates the structure relations S_d and S_e with the proximal and regionally proximal relations of (X, T). Theorem 3 says that these four relations all coincide if (X, T) is locally almost periodic. The concluding remarks show how any transformation group with compact phase space and noncompact phase group gives rise in a natural way to a compact topological group, called its *structure group*. Such transformation groups, in particular minimal sets, are thus partially classifiable according to their structure groups. As a general reference for the notions occurring here, consult [4].

Definition 1. Let (X, T, π) and (Y, T, ρ) be transformation groups with the same phase group T. A *homomorphism of* (X, T, π) *into* or *onto* (Y, T, ρ) is defined to be a continuous map ϕ of X into or onto Y such that $t \in T$ implies $\pi^t\phi = \phi\rho^t$, or in the condensed notation, such that $x \in X$ and $t \in T$ implies $xt\phi = x\phi t$. A homomorphism which is at the same time a homeomorphism is called an *isomorphism*. Of course, any intrinsic property of transformation groups is preserved under isomorphisms onto. See [4, 12.51 and 12.54] for a nontrivial example of a homomorphism taken from symbolic dynamics.

Standing notation. Throughout (X, T, π) and (Y, T, ρ) will be transformation groups with compact (bicompact Hausdorff) phase spaces X and Y, and ϕ will be a homomorphism of (X, T) onto (Y, T).

Definition 2. If $f: M \to N$, then $\hat{f}: M \times M \to N \times N$ is defined by $(m_1, m_2)\hat{f} = (m_1 f, m_2 f)$ for $(m_1, m_2) \in M \times M$; when no ambiguity is possible, we sometimes write $(m_1, m_2)f$ instead of $(m_1, m_2)\hat{f}$. If $F \subset N^M$, then $\hat{F}$ denotes $[\hat{f} | f \in F]$; when no ambiguity is possible, we sometimes write AF in place of $A\hat{F} = [a\hat{f} | a \in A \text{ and } f \in F]$ where $A \subset M \times M$.

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REMARK 1. The map $\hat{\phi}$ is a homomorphism of $(X\times X, T)$ onto $(Y\times Y, T)$, where $(X\times X, T)$ denotes the squared transformation group defined by the condition that $(x_1, x_2)\in X\times X$ and $t\in T$ imply $(x_1, x_2)t=(x_1t, x_2t)$.

REMARK 2. Let R be an invariant closed equivalence relation in X, and let λ be the canonical map of X onto $X|R$. Then λ is a homomorphism of (X, T) onto $(X|R, T)$, where $(X|R, T)$ denotes the partition transformation group defined by the condition that $x\in X$ and $t\in T$ imply $(xR)t=(xt)R$. That R is invariant closed means R is an invariant closed subset of the phase space of $(X\times X, T)$. As usual $X|R$ denotes the partition $[xR|x\in X]$ of X provided with the quotient topology.

REMARK 3. Let $R=\Delta_Y\phi^{-1}$, and let $\mu: Y\to X|R$ be defined by $y\mu=y\phi^{-1}$ $(y\in Y)$. Then μ is an isomorphism of (Y, T) onto $(X|R, T)$. Here Δ_Y denotes the diagonal of $Y\times Y$. This remark, together with Remark 2, shows that homomorphic images of transformation groups and partition transformation groups are simply different ways of speaking of the same thing. For convenience, we shall sometimes use one language rather than the other.

REMARK 4. Let R_1 and R_2 be invariant closed equivalence relations in X such that $R_1\subset R_2$, and let λ be the canonical map of $X|R_1$ onto $X|R_2$. Then λ is a homomorphism of $(X|R_1, T)$ onto $(X|R_2, T)$.

REMARK 5. Let $((X_i, T)|i\in I)$ be a family of transformation groups with the same phase group T, for each $i\in I$ let λ_i be a homomorphism of (X, T) into (X_i, T) such that $(\lambda_i|i\in I)$ separates points of X, and let $\lambda: X\to \times_{i\in I} X_i$ be the map such that $x\in X$ implies $x\lambda=(x\lambda_i|i\in I)$. Then λ is an isomorphism of (X, T) into $(\times_{i\in I} X_i, T)$, where $(\times_{i\in I} X_i, T)$ denotes the cartesian product transformation group defined by the condition that $(x_i|i\in I)\in \times_{i\in I} X_i$ and $t\in T$ imply $(x_i|i\in I)t=(x_it|i\in I)$.

DEFINITION 3. The transformation group (X, T) is said to be *distal* in case $x_1, x_2\in X$ with $x_1\neq x_2$ implies the existence of an index α of X such that $(x_1t, x_2t)\notin\alpha$ for all $t\in T$. The space X, being compact, has a unique compatible uniformity. The transformation group (X, T) is said to be *equicontinuous* or *uniformly equicontinuous*, in case the transition group $G=[\pi^t|t\in T]$ has this property. It is readily proved that uniformly equicontinuous implies distal, but not conversely. (Consider a quasirotation of an annulus, the individual circles spinning at different rates.) It is known [4, 4.38] that these three properties of (X, T) are pairwise equivalent: equicontinuous, uniformly equicontinuous, almost periodic.

REMARK 6. If (X, T) is [distal] [equicontinuous] and if Z is an invariant subset of X, then (Z, T) is [distal] [equicontinuous].

REMARK 7. Let $((X_i, T)|i\in I)$ be a family of [distal] [equicontinuous] transformation groups with the same phase group. Then $(\times_{i\in I} X_i, T)$ is also [distal] [equicontinuous].

LEMMA 1. *Let I be a set, for each $i\in I$ let R_i be an invariant closed equivalence* *relation in* X *such that* T *is [distal] [equicontinuous] on* $X|R_i$, *and let* $R=\bigcap_{i\in I} R_i$. *Then* R *is an invariant closed equivalence relation in* X *such that* T *is [distal] [equicontinuous] on* $X|R$.

Proof. It is readily verified that R is an invariant closed equivalence relation in X. By Remark 4, for each $i\in I$ the canonical map λ_i of $X|R$ onto $X|R_i$ is a homomorphism of $(X|R, T)$ onto $(X|R_i, T)$. By the definition of R, $(\lambda_i|i\in I)$ separates points of $X|R$. By Remark 5, there exists an isomorphism λ of $(X|R, T)$ into $(\times_{i\in I} X|R_i, T)$. By Remark 7, $(\times_{i\in I} X|R_i, T)$ is [distal] [equicontinuous]. By Remark 6, $((X|R)\lambda, T)$ is [distal] [equicontinuous]. Hence, $(X|R, T)$ is [distal] [equicontinuous].

THEOREM 1. *There exists a least invariant closed equivalence relation* S *in* X *such that* T *is [distal] [equicontinuous] on* $X|S$.

Proof. Let $(R_i|i\in I)$ be the family of all invariant closed equivalence relations R in X such that T is [distal] [equicontinuous] on $X|R$. Such exist; for example, take $R=X\times X$. Define $S=\bigcap_{i\in I} R_i$. The conclusion follows from Lemma 1.

DEFINITION 4. The least invariant closed equivalence relation S in X such that T is distal on $X|S$ is called the *distal structure relation of* (X, T) and the corresponding transformation group $(X|S, T)$ is called the *distal structure transformation group of* (X, T). We may denote the distal structure relation of (X, T) by S (distal; (X, T)) or S (distal; X) or S_X (distal) or S (distal). Completely analogous definitions hold if "distal" is replaced by "equicontinuous." Clearly, S (distal) $\subset S$ (equicontinuous); the inclusion may be proper. [Consider a quasirotation.]

REMARK 8. Our primary concerns in this paper are the distal and equicontinuous structure relations. However, we observe that the properties distal and equicontinuous may be replaced in Theorem 1 and Definition 4 by any other property which satisfies Remarks 6 and 7.

EXAMPLE 1. Let A, B, C be concentric circles in the plane with radii 1, 2, 3. Let X be the union of A, B, C together with a counterclockwise spiral from A to B as limit sets and a counterclockwise spiral from B to C as limit sets. Let C be provided with the topology induced by that of the plane. Let $T=\mathcal{I}$, the additive group of all integers with its discrete topology, and let π^1 be the homeomorphism of X onto X which is a spin of 1 radian on A, B, C and which increases angular coordinates by 1 radian for points on the spiral.

DEFINITION 5. The points x and y of X are said to be *proximal* (each *to* the other) provided that if α is an index of X, then there exists $t\in T$ such that $(xt, yt)\in\alpha$. The *proximal relation of* (X, T), denoted $P(X, T)$ or $P(X)$ or P_X or P, is defined to be the set of all couples $(x, y)\in X\times X$ such that x is proximal to y. We observe $P=\bigcap_{\alpha\in\mathfrak{U}} \alpha T$ where $\mathfrak{U}$ is the uniformity of X. The proximal relation P is invariant reflexive symmetric but not necessarily transitive nor closed [see Example 1]. The points x and y of X are said to be *distal* (each *from* the other) provided that x is not proximal to y. The transformation group (X, T) is distal if and only if $P=\Delta$, that is, every two different points of X are distal.

DEFINITION 6. The points x and y of X are said to be *regionally proximal* (each *to* the other) provided that if U, V are neighborhoods of x, y, and if α is an index of X, then there exist $x_1 \in U$, $y_1 \in V$, and $t \in T$ such that $(x_1t, y_1t) \in \alpha$. *The regionally proximal relation of* (X, T), denoted $Q(X, T)$ or $Q(X)$ or Q_X or Q, is defined to be the set of all couples $(x, y) \in X \times X$ such that x is regionally proximal to y. We observe $Q = \bigcap_{\alpha \in \mathcal{U}} [\alpha T]^-$ where $\mathcal{U}$ is the uniformity of X. The regionally proximal relation Q is invariant closed reflexive symmetric but not necessarily transitive [see Example 1]. Clearly $P \subset Q$; the inclusion may be proper [see Example 1]. The points x and y of X are said to be *regionally distal* (each *from* the other) provided that x is not regionally proximal to y. The transformation group (X, T) is equicontinuous if and only if $Q=\Delta$, that is, every two different points of X are regionally distal. Note that the (uniform) equicontinuity of (X, T) is equivalent to the condition that for each $\alpha \in \mathcal{U}$ there exist $\beta \in \mathcal{U}$ such that $\beta T \subset \alpha$.

DEFINITION 7. If $t \in T$, then $\pi^t \colon X \to X$ is a homeomorphism of X onto X which is called the *transition of* (X, T) *induced by* t. The set $[\pi^t \mid t \in T]$ of all transitions of (X, T) is a group of homeomorphisms of X onto X, which is called the *transition group of* (X, T), and which is denoted by $G(X, T)$ or $G(X)$ or G_X or G.

REMARK 9. Let X^X denote as usual the cartesian power with base X and exponent X, that is, X^X is the set of all maps of X into X. Let X^X be provided with its cartesian product topology and its cartesian product uniformity, or what is equivalent, with its point-open topology and its point-index uniformity. Convergence in X^X thus means pointwise convergence on X. The theorem of Tychonoff tells us that X^X is compact. Now X^X is also a multiplicative semigroup. Even though not all elements of X^X are continuous, it has turned out [2] that the compact semigroup X^X is useful in the study of the transformation group (X, T). Moreover, the multiplication in X^X is in general not continuous but has certain notable continuity properties. They are: each left multiplication in X^X is continuous on X^X, and each right multiplication in X^X with multiplier continuous on X is continuous on X^X. This may be proved as follows: If $\xi, \eta \in X^X$, if $(\eta_i \mid i \in I)$ is a net in X^X such that $\eta_i \to \eta$, and if $x \in X$, then $x\xi\eta_i \to x\xi\eta$ always and $x\eta_i\xi \to x\eta\xi$ whenever ξ is continuous on X. In particular, each right multiplication in X^X with multiplier in G is continuous on X^X.

DEFINITION 8. *The enveloping semigroup of* (X, T) denoted $E(X, T)$ or $E(X)$ or E_X or E, is defined to be the closure $\overline{G}$ in X^X of the transition group G of (X, T). The enveloping semigroup E of (X, T) is thus a compact semigroup in X^X. Clearly, E is compact. To see that E is a semigroup, argue as follows. If $\eta \in G$, then $E\eta = \overline{G}\eta = [G\eta]^- = \overline{G} = E$ since G is a group and right multiplication by η is continuous. Thus $EG \subset E$. If $\xi \in E$, then $\xi E = \xi \overline{G}$

$=[\xi G]^- \subset \bar{E} = E$ by the preceding statement and the fact that left multiplication by ξ is continuous. Thus $EE \subset E$. Each left multiplication in E is continuous on E, and each right multiplication in E with multiplier in G is continuous on E.

LEMMA 2. *The following statements hold:*

(1) *If $x \in X$ and if T is recursive at x, then T is recursive at $x\phi$.*

(2) *If T is pointwise recursive on X, then T is pointwise recursive on Y.*

(3) *If T is recursive on X, then T is recursive on Y.*

(4) *If T is distal on X, then T is distal on Y.*

(5) *If T is equicontinuous on X, then T is equicontinuous on Y.*

(6) *If $x \in X$, then $[xT]^-\phi = [x\phi T]^-$.*

(7) *If X is minimal under T, then Y is minimal under T.*

(8) *$P_X\hat{\phi} \subset P_Y$; the inclusion may be proper.*

(9) *$Q_X\hat{\phi} \subset Q_Y$; the inclusion may be proper.*

(10) *There exists a unique map ψ of G_X into G_Y such that $t \in T$ implies $\pi^t\psi = \rho^t$. The map ψ is a uniformly continuous group homomorphism of G_X onto G_Y such that*

$$\xi \in G_X \text{ implies } \xi \circ \phi = \phi \circ (\xi\psi).$$

(11) *There exists a unique map θ of E_X into E_Y such that θ is continuous and θ extends ψ. The map θ is a uniformly continuous semigroup homomorphism of E_X onto E_Y such that $\xi \in E_X$ implies $\xi \circ \phi = \phi \circ (\xi\theta)$.*

Proof. (1) Let V be a neighborhood of $x\phi$. Choose a neighborhood U of x such that $U\phi \subset V$, and choose an admissible subset A of T such that $xA \subset U$. Then $x\phi A = xA\phi \subset U\phi \subset V$.

(2) Since ϕ maps X onto Y, statement (2) follows immediately from statement (1).

(3) Let β be an index of Y. Since ϕ is uniformly continuous, there exists an index α of X such that $x \in X$ implies $x\alpha\phi \subset x\phi\beta$. Choose an admissible subset A of T such that $x \in X$ implies $xA \subset x\alpha$. Then $x \in X$ implies $x\phi A = xA\phi \subset x\alpha\phi \subset x\phi\beta$. Hence $y \in Y$ implies $yA \subset y\beta$.

(4) It is known [2, Theorem 1] that for a transformation group (X, T) with compact phase space, T is distal on X if and only if T is pointwise almost periodic on $X \times X$. Since $\hat{\phi}$ is a homomorphism of the transformation group $(X \times X, T)$ onto the transformation group $(Y \times Y, T)$, the desired conclusion follows from (2).

(5) It is known [4, 4.38] that for a transformation group (X, T) with compact phase space, T is equicontinuous on X if and only if T is almost periodic on X. The desired conclusion now follows from (3).

(6) If $x \in X$, then $[xT]^-\phi = [xT\phi]^- = [x\phi T]^-$.

(7) If $x \in X$, then $[x\phi T]^- = [xT]^-\phi = X\phi = Y$.

(8) Let $(x_1, x_2) \in P_X$. Let β be an index of Y. Choose an index α of X so

that $\alpha\hat{\phi}\subset\beta$. There exists $t\in T$ such that $(x_1t, x_2t)\in\alpha$. Then $(x_1\phi, x_2\phi)t = (x_1\phi t, x_2\phi t) = (x_1t\phi, x_2t\phi) = (x_1t, x_2t)\hat{\phi}\in\alpha\hat{\phi}\subset\beta$. Hence $(x_1, x_2)\hat{\phi}\in P_Y$. See Example 2.

(9) Let $(x_1, x_2)\in Q_X$. Let V_1, V_2 be neighborhoods of $x_1\phi$, $x_2\phi$, and let β be an index of Y. Choose neighborhoods U_1, U_2 of x_1, x_2 and an index α of X so that $U_1\phi\subset V_1$, $U_2\phi\subset V_2$, and $\alpha\hat{\phi}\subset\beta$. There exist $z_1\in U_1$, $z_2\in U_2$, and $t\in T$ such that $(z_1t, z_2t)\in\alpha$. Then $z_1\phi\in U_1\phi\subset V_1$, $z_2\phi\in U_2\phi\subset V_2$, and $(z_1\phi, z_2\phi)t = (z_1t, z_2t)\hat{\phi}\in\alpha\hat{\phi}\subset\beta$. Hence $(x_1, x_2)\hat{\phi}\in Q_Y$. See Example 2.

(10) Let $t, s\in T$ such that $\pi^t=\pi^s$. We show $\rho^t=\rho^s$. If $x\in X$, then $x\phi\rho^t = x\pi^t\phi = x\pi^s\phi = x\phi\rho^s$. Since ϕ is onto Y, it follows that $y\in Y$ implies $y\rho^t=y\rho^s$, whence $\rho^t=\rho^s$. The unique existence of ψ is now proved.

The map ψ is clearly onto G_Y. The map ψ is a group homomorphism since $t, s\in T$ implies $(\pi^t\circ\pi^s)\psi = \pi^{ts}\psi = \rho^{ts} = \rho^t\circ\rho^s = (\pi^t\psi)\circ(\pi^s\psi)$. Now $t\in T$ implies $\pi^t\circ\phi = \phi\circ\rho^t = \phi\circ\pi^t\psi$. That is to say, $\xi\in G_X$ implies $\xi\circ\phi = \phi\circ\xi\psi$.

We show ψ is uniformly continuous. Let N be a finite set, for each $n\in N$ let $y_n\in Y$, and let β be an index of Y. For each $n\in N$ choose $x_n\in X$ so that $x_n\phi = y_n$, and choose an index α of X so that $\alpha\hat{\phi}\subset\beta$. Let $t, s\in T$ such that $n\in N$ implies $(x_n\pi^t, x_n\pi^s)\in\alpha$. Then $n\in N$ implies $(y_n(\pi^t\psi), y_n(\pi^s\psi)) = (y_n\rho^t, y_n\rho^s) = (x_n\phi\rho^t, x_n\phi\rho^s) = (x_n\pi^t\phi, x_n\pi^s\phi)\in\alpha\hat{\phi}\subset\beta$. The proof is completed.

(11) Since G_X is dense in E_X and E_Y is a complete separated (indeed, compact) uniform space, and $\psi\colon G_X\to E_Y$ is uniformly continuous, the unique existence of θ is assured. That θ is uniformly continuous follows either from the uniform continuity of ψ or the compactness of E_X. Since E_X is compact and hence $\overline{G}_X\theta$ is closed, we have $E_X\theta = \overline{G}_X\theta\supset[G_X\theta]^- = \overline{G}_Y = E_Y$.

Let $\xi\in E_X$. Choose a net $(\xi_i \mid i\in I)$ in G_X so that $\xi_i\to\xi$. If $x\in X$, then $x\xi_i\to x\xi$, $(x\xi_i)\phi\to(x\xi)\phi = x(\xi\circ\phi)$, $(x\xi_i)\phi = x(\xi_i\circ\phi) = x(\phi\circ(\xi_i\psi)) = (x\phi)(\xi_i\psi) = (x\phi)(\xi_i\theta)\to(x\phi)(\xi\theta) = x(\phi\circ(\xi\theta))$, and $x(\xi\circ\phi) = x(\phi\circ(\xi\theta))$. Hence $\xi\circ\phi = \phi\circ(\xi\theta)$.

Let $\xi\in E_X$ and $\eta\in G_X$. We show $(\xi\circ\eta)\theta = \xi\theta\circ\eta\theta$. Choose a net $(\xi_i \mid i\in I)$ in G_X so that $\xi_i\to\xi$. Then $\xi_i\circ\eta\to\xi\circ\eta$ and $(\xi_i\circ\eta)\theta\to(\xi\circ\eta)\theta$. Also $(\xi_i\circ\eta)\theta = (\xi_i\circ\eta)\psi = \xi_i\psi\circ\eta\psi = \xi_i\theta\circ\eta\psi\to\xi\theta\circ\eta\psi = \xi\theta\circ\eta\theta$. Hence $(\xi\circ\eta)\theta = \xi\theta\circ\eta\theta$.

Let $\xi\in E_X$ and $\eta\in E_X$. We show $(\xi\circ\eta)\theta = \xi\theta\circ\eta\theta$. Choose a net $(\eta_i \mid i\in I)$ in G_X so that $\eta_i\to\eta$. Then $\xi\circ\eta_i\to\xi\circ\eta$ and $(\xi\circ\eta_i)\theta\to(\xi\circ\eta)\theta$. Also $(\xi\circ\eta_i)\theta = \xi\theta\circ\eta_i\theta\to\xi\theta\circ\eta\theta$. Hence $(\xi\circ\eta)\theta = \xi\theta\circ\eta\theta$.

EXAMPLE 2. Let $X = [a \mid 0\leqq a\leqq 1]\times[0, 1]$, let X be provided with the topology induced by the natural topology of the plane, let $T=\mathcal{I}$ where $\mathcal{I}$ denotes the additive group of all integers with its discrete topology, and let $(a, 0)\pi^1 = (a^2, 0)$ and $(a, 1)\pi^1 = (a^2, 1)$ $(0\leqq a\leqq 1)$. Let Y be the space obtained from X by identification of the points $(0, 0)$ and $(0, 1)$, let ϕ be the natural map of X onto X, and let ρ be defined in the evident way so that ϕ is a homomorphism of (X, T, π) onto (Y, T, ρ).

Standing notation. Throughout ψ will denote the canonical map of G_X onto G_Y and θ will denote the canonical map of E_X onto E_Y induced by ϕ

according to (10) and (11) of Lemma 2. Note that ψ and θ do not depend upon ϕ itself but are uniquely defined once the existence of a homomorphism of (X, T) onto (Y, T) is established.

REMARK 10. Let R_1 and R_2 be invariant closed equivalence relations in X, such that $R_1 \subset R_2$. If T is [distal] [equicontinuous] on $X|R_1$ then T is [distal] [equicontinuous] on $X|R_2$.

REMARK 11. Let R be an invariant closed equivalence relation in X. Then: (1) T is distal on $X|R$ if and only if $R \supset S$ (distal); (2) T is equicontinuous on $X|R$ if and only if $R \supset S$ (equicontinuous).

REMARK 12. (1) S (distal)$\supset P$; the inclusion may be proper. (2) S (equicontinuous)$\supset Q$; the inclusion may be proper [see Example 1].

LEMMA 3. *Let M be a nonvacuous compact Hausdorff space with a multiplicative semigroup structure such that all left multiplications in M are continuous on M. Then*: (1) *M contains at least one idempotent.* (2) *If M has a right unity element e, then M is a group if and only if e is the only idempotent in M.*

Proof. (1) See [2, Lemma 1].

(2) The necessity is clear. We prove the sufficiency. Let $a \in M$. Then aM is a nonvacuous compact subset of M, and aM is semigroup since $(aM)(aM) \subset (aM)M = a(MM) \subset aM$. By (1) there exists an idempotent u in aM. Thus $u = ab$ for some $b \in M$. Since $u = e$ by hypothesis, we have shown that a has a right inverse, namely b. The proof is completed.

LEMMA 4. *If $x, y \in X$, then $(x, y) \in P$ if and only if $x\xi = y\xi$ for some $\xi \in E$. In symbols, $P = \Delta E^{-1}$.*

Proof. Suppose $x\xi = y\xi$ for some $\xi \in E$. Let α be an index of X. Choose a neighborhood U of $x\xi = y\xi$ so that $U \times U \subset \alpha$. There exists $\eta \in G$ such that $x\eta \in U$ and $y\eta \in U$ whence $(x\eta, y\eta) \in U \times U \subset \alpha$. Thus $(x, y) \in P$.

Suppose $(x, y) \in P$. Let $\mathfrak{U}$ be the uniformity of X. For each $\alpha \in \mathfrak{U}$ define $F_\alpha = [\eta | \eta \in G \text{ and } (x\eta, y\eta) \in \alpha]$. Define $\mathfrak{F} = [F_\alpha | \alpha \in \mathfrak{U}]$. Now $\mathfrak{F}$ is a filter base on the compact space E and hence there exists $\xi \in \bigcap \overline{\mathfrak{F}}$. Assume $x\xi \neq y\xi$. Choose a symmetric index α of X so that $(x\xi, y\xi) \notin \alpha^3$. Since $\xi \in \overline{F}_\alpha$, there exists $\eta \in F_\alpha$ such that $x\eta \in x\xi\alpha$ and $y\eta \in y\xi\alpha$. Therefore

$$(x\xi, y\xi) = (x\xi, x\eta)(x\eta, y\eta)(y\eta, y\xi) \in \alpha^3$$

which is a contradiction. Hence $x\xi = y\xi$.

LEMMA 5. *The transformation group (Y, T) is distal if and only if $P_X\hat{\phi} = \Delta_Y$.*

Proof. Suppose $P_X\hat{\phi} = \Delta_Y$. It is known [2, Theorem 1] that (Y, T) is distal if and only if E_Y is a group. By 2 of Lemma 3, E_Y is a group if and only if the identity map of Y is the only idempotent in E_Y. Let v be an idempotent in E_Y. The proof that (Y, T) is distal will be completed when we show that v is the identity map of Y. By (11) of Lemma 2, $v\theta^{-1}$ is a nonvacuous compact semigroup in E_X. By (1) of Lemma 3 there exists an idempotent u in $v\theta^{-1}$. By Lemma 4, $(xu, x)\in P_X$ for all $x\in X$ since $(xu)u=xu^2=xu$. By hypothesis, therefore, $xu\phi=xu$. By (11) of Lemma 2, $x\in X$ implies $x\phi v=xu\phi=x\phi$. Hence $y\in Y$ implies $yv=y$, and v is the identity map of Y.

Suppose (Y, T) is distal. Using (8) of Lemma 2 it follows that $\Delta_Y=\Delta_X\hat{\phi}\subset P_X\hat{\phi}\subset P_Y=\Delta_Y$. Hence $P_X\hat{\phi}=\Delta_Y$.

DEFINITION 9. Let M and N be topological spaces, and let $f\colon M\to N$. For $x\in M$, the *oscillation of f at x, denoted* $\omega(f; x)$, is defined to be $[xf]\times\bigcap[\mathfrak{N}_xf]^-$, which is a subset of $N\times N$. The symbol $\mathfrak{N}_x$ denotes the neighborhood filter of $x\in M$, and $\bigcap[N_xf]^-$ is the adherence of the filter base $\mathfrak{N}_xf$ on N. The *oscillation of f*, denoted $\omega(f)$, is defined to be $\bigcup_{x\in M}\omega(f; x)$. If F is a set of maps of M into N, then the *oscillation of F*, denoted $\omega(F)$, is defined to be $\bigcup_{f\in F}\omega(f)$. Suppose N is regular. We observe that if f is continuous at x, then $\omega(f; x)\subset\Delta_N$; if f is continuous on M, then $\omega(f)\subset\Delta_N$; if each member of F is continuous on M, then $\omega(F)\subset\Delta_N$; the converses all hold if N is compact. These notions are closely related to similar notions in real analysis. Just as in the classical case, oscillation is a measure of discontinuity.

DEFINITION 10. The *oscillation of* (X, T), denoted $\Omega(X, T)$ or Ω_X or Ω, is defined to be the oscillation $\omega(E)$ of the enveloping semigroup E of (X, T). To summarize,

$$\Omega=\omega(E)=\bigcup_{\xi\in E}\omega(\xi)=\bigcup_{\xi\in E}\bigcup_{x\in X}\omega(\xi; x)=\bigcup_{\xi\in E}\bigcup_{x\in X}([x\xi]\times\bigcap[\mathfrak{N}_x\xi]^-).$$

Every member of E is continuous on X if and only if $\Omega\subset\Delta$, or equivalently $\Omega=\Delta$, since always $\Omega\supset\Delta$.

LEMMA 6. *The following statements hold*:

(1) $\omega(\xi, x)\hat{\phi}\subset\omega(\xi\theta, x\phi)$ *if* $\xi\in E_X$ *and* $x\in X$.

(2) $\omega(\xi)\hat{\phi}\subset\omega(\xi\theta)$ *if* $\xi\in E_X$.

(3) $\omega(E_X)\hat{\phi}\subset\omega(E_Y)$; *or equivalently*, $\Omega_X\hat{\phi}\subset\Omega_Y$.

Proof. (1) If $\xi\in E_X$ and if $x\in X$, then

$$\begin{aligned}\omega(\xi; x)\hat{\phi} &= ([x\xi]\times\bigcap[\mathfrak{N}_x\xi]^-)\hat{\phi}=[x\xi\phi]\times(\bigcap[\mathfrak{N}_x\xi]^-)\phi=[x\xi\phi]\times\bigcap[\mathfrak{N}_x\xi\phi]^- \\ &= [(x\phi)(\xi\theta)]\times\bigcap[\mathfrak{N}_x\phi(\xi\theta)]^-\subset[(x\phi)(\xi\theta)]\times\bigcap[\mathfrak{N}_{x\phi}(\xi\theta)]^- \\ &= \omega(\xi\theta; x\phi)\end{aligned}$$

since $\mathfrak{N}_x\phi\supset\mathfrak{N}_{x\phi}$.

(2) By (1),

$$\begin{aligned}\omega(\xi)\hat{\phi} &= \left(\bigcup_{x\in X}\omega(\xi; x)\right)\hat{\phi}=\bigcup_{x\in X}\omega(\xi; x)\hat{\phi}\subset\bigcup_{x\in X}\omega(\xi\theta; x\phi) \\ &= \bigcup_{y\in Y}\omega(\xi\theta; y)=\omega(\xi\theta).\end{aligned}$$

(3) By (2),

$$\Omega_X\hat{\phi} = \omega(E_X)\hat{\phi} = (\bigcup [\omega(\xi) \mid \xi \in E_X])\hat{\phi} = \bigcup [\omega(\xi)\hat{\phi} \mid \xi \in E_X] \subset \bigcup [\omega(\xi\theta) \mid \xi \in E_X]$$
$$= \bigcup [\omega(\eta) \mid \eta \in E_Y] = \omega(E_Y) = \Omega_Y.$$

LEMMA 7. *Let M and N be topological spaces, let $f\colon M \to N$, and let $x \in M$. The following statements are equivalent*: (1) *f is continuous at x.* (2) *If $\mathfrak{F}$ is an ultrafilter on M such that $\mathfrak{F}f \to x$, then $\mathfrak{F}f \to xf$.*

Proof. Clearly (1) implies (2). Assume (2). We prove (1). We suppose without loss that $Mf = N$.

If $\mathfrak{F}$ is a filter on a set A, then $\mathfrak{F}^*$ denotes the collection of all ultrafilters on A which contain $\mathfrak{F}$; whence $\bigcap \mathfrak{F}^* = \mathfrak{F}$ and if λ is a map of A onto a set B, then $\mathfrak{F}^*\lambda = (\mathfrak{F}\lambda)^*$.

Consequently, $\mathfrak{N}_{xf} \subset \bigcap \mathfrak{N}_x^* f) = \bigcap (\mathfrak{N}_x f)^* = \mathfrak{N}_x f$, $\mathfrak{N}_{xf} \subset \mathfrak{N}_x f$, and f is continuous at x.

LEMMA 8. *Every member of E_Y is continuous on Y if and only if $\Omega_X\hat{\phi} = \Delta_Y$*

Proof. If every member of E_Y is continuous on Y, it follows from (3) of Lemma 6 that $\Delta_Y = \Delta_X\hat{\phi} \subset \Omega_X\hat{\phi} \subset \Omega_Y = \Delta_Y$ whence $\Omega_X\hat{\phi} = \Delta_Y$. We prove the sufficiency.

Let $\eta \in E_Y$, let $y \in Y$, and let $\mathfrak{G}$ be an ultrafilter on Y such that $\mathfrak{G} \to y$. By Lemma 7 it is enough to show that $\mathfrak{G}\eta \to y\eta$. By (11) of Lemma 2, $\xi\theta = \eta$ for some $\xi \in E_X$. Choose an ultrafilter $\mathfrak{F}$ on X so that $\mathfrak{F} \supset \mathfrak{G}\phi^{-1}$ whence $\mathfrak{F}\phi = \mathfrak{G}$. Now $\mathfrak{F} \to x_1$, for some $x_1 \in X$, and $\mathfrak{F}\xi \to x_2$ for some $x_2 \in X$ since $\mathfrak{F}\xi$ is an ultrafilter base on X. From $\mathfrak{F}\phi \to x_1\phi$ and $\mathfrak{F}\phi = \mathfrak{G} \to y$ we conclude that $x_1\phi = y$. Now $(x_1\xi,\ x_2) \in [x_1\xi] \times \bigcap [\mathfrak{F}\xi]^- \subset [x_1\xi] \times \bigcap [\mathfrak{N}_{x_1}\xi]^- = \omega(\xi;\ x_1) \subset \Omega_X$. By hypothesis $(x_1\xi\phi,\ x_2\phi) = (x_1\xi,\ x_2)\hat{\phi} \in \Omega_X\hat{\phi} = \Delta_Y$ whence $x_1\xi\phi = x_2\phi$. Using (11) of Lemma 2 it follows that $\mathfrak{G}\eta = \mathfrak{F}\phi\eta = \mathfrak{F}\xi\phi \to x_2\phi = x_1\xi\phi = x_1\phi\eta = y\eta$. Thus $\mathfrak{G}\eta \to y\eta$ and the proof is completed.

LEMMA 9. $\Omega \subset Q$; *the inclusion may be proper.*

Proof. Let $x \in X$, let $\xi \in E$, and let $y \in \bigcap [\mathfrak{N}_x\xi]^-$. We show $(x\xi,\ y) \in Q$. Let U, V be open neighborhoods of $x\xi$, y and let α be an index of X. Choose a neighborhood W of x so that $W \times W \subset \alpha$. Since $y \in [W\xi]^-$, there exists $a \in W$ such that $a\xi \in V$. Choose $\eta \in G$ so that $x\eta \in U$ and $a\eta \in V$. Then $(x\eta,\ a\eta)\eta^{-1} = (x,\ a) \in \alpha$. Hence $(x\xi,\ y) \in Q$. The proof is completed by an inspection of Example 3.

EXAMPLE 3. Let T be $\mathscr{I}$ or $\mathscr{R}$ where $\mathscr{I}$ is the additive group of all integers with its discrete topology and $\mathscr{R}$ is the additive group of all real numbers with its natural topology, let $X = T \cup [\infty]$ be the one-point compactification of the underlying space of T, and let $\pi\colon X \times T \to X$ be the continuous extension of the group addition in T where $\infty\pi^t = \infty$ for all $t \in T$. For the transformation group $(X,\ T,\ \pi)$ we have $E = G \cup [\lambda_\infty]$ where $\lambda_\infty\colon X \to X$ is the constant map which maps all of X into $[\infty]$, $\Omega = \Delta$, and $P = Q = X \times X$.

LEMMA 10. *If* $Q_X\hat{\phi}=\Delta_Y$, *then* E_Y *is a group of homeomorphisms of* Y *onto* Y.

Proof. Since $\Delta_X\subset P_X\subset Q_X$, we have $P_X\hat{\phi}=\Delta_Y$. By Lemma 5, (Y, T) is distal. Hence, by [2, Theorem 1], E_Y is a group of permutations of Y. Since $\Delta_X\subset\Omega_X\subset Q_X$ by Lemma 9, we have $\Omega_X\hat{\phi}=\Delta_Y$. By Lemma 8, every member of E_Y is continuous. The proof is completed.

LEMMA 11. *The transformation group* (X, T) *is equicontinuous if and only if* E *is a group of homeomorphisms of* X *onto* X.

Proof. Lemma 11 is a corollary of [1, Theorem 3].

LEMMA 12. *The transformation group* (Y, T) *is equicontinuous if and only if* $Q_X\hat{\phi}=\Delta_Y$.

Proof. The sufficiency follows from Lemmas 10 and 11. Suppose (Y, T) is equicontinuous. Using (9) of Lemma 2 it follows that $\Delta_Y=\Delta_X\hat{\phi}\subset Q_X\hat{\phi}\subset Q_Y=\Delta_Y$ [see Definition 6]. Hence $Q_X\hat{\phi}=\Delta_Y$.

THEOREM 2. *The following statements hold*:

(1) *If* R *is an invariant closed equivalence relation in* X, *then* T *is* [*distal*] [*equicontinuous*] *on* $X|R$ *if and only if* R *contains the* [*proximal*] [*regionally proximal*] *relation of* (X, T).

(2) *The* [*distal*] [*equicontinuous*] *structure relation of* (X, T) *coincides with the least invariant closed equivalence relation in* X *which contains the* [*proximal*] [*regionally proximal*] *relation of* (X, T).

Proof. (1) Let R be an invariant closed equivalence relation in X, and let $\lambda: X\rightarrow X|R$ be the canonical map of X onto $X|R$. Now λ is a homomorphism of (X, T) onto $(X|R, T)$. By Lemma 5, $(X|R, T)$ is distal if and only if $P_X\hat{\lambda}=\Delta(X|R)$. Since $P_X\hat{\lambda}=\Delta(X|R)$ if and only if $R\supset P_X$, we have that $(X|R, T)$ is distal if and only if $R\supset P_X$. By Lemma 12, $(X|R, T)$ is equicontinuous if and only if $Q_X\hat{\lambda}=\Delta(X|R)$. Since $Q_X\hat{\lambda}=\Delta(X|R)$ if and only if $R\supset Q_X$, we have that $(X|R, T)$ is equicontinuous if and only if $R\supset Q_X$.

(2) This follows immediately from (1) and the definition of [distal] [equicontinuous] structure relation of (X, T).

DEFINITION 11. The transformation group (X, T) is said to be *locally almost periodic* provided that if $x\in X$ and if U is a neighborhood of x, then there exist a neighborhood V of x and a syndetic subset A of T such that $VA\subset U$.

DEFINITION 12. Subsets Y and Z of X are said to be *distal* (each *from* the other) provided there exists an index α of X such that $y\in Y$, $z\in Z$ and $t\in T$ implies $(yt, zt)\notin\alpha$.

LEMMA 13. *Let* T *be locally almost periodic on* X. *Then*:

(1) *If* $x, y\in X$ *such that* x *is proximal to* y, *and if* α *is an index of* X, *then there exists a syndetic subset* A *of* T *such that that* $(xa, ya)\in\alpha$ *for all* $a\in A$.

(2) *If* $x\in X$, *if* $Z\subset X$, *and if* x *is distal from* Z, *then there exists a neighborhood* U *of* x *such that* U *is distal from* Z.

(3) *If* $x, y\in X$ *such that* x *is distal from* y, *then there exists neighborhoods* U *of* x *and* V *of* y *such that* U *is distal from* V.

(4) $P=Q$ *and* P *is an invariant closed equivalence relation in* X.

Proof. (1) Let U be a neighborhood of x such that $U\times U\subset\alpha$. There exists an index β of X and a syndetic subset B of T such that $x\beta^2B\subset U$. There exists a syndetic subset C of T such that $xC\subset x\beta$. Let K be a compact subset of T such that $T=CK$. Choose an index γ of X so that $(x_1, x_2)\in\gamma$ implies $(x_1k^{-1}, x_2k^{-1})\in\beta$ for all $k\in K$. There exists $t\in T$ such that $(xt, yt)\in\gamma$. Write $t=ck$ where $c\in C$ and $k\in K$. Then $(xc, yc)=(xtk^{-1}, ytk^{-1})\in\beta$. Now $xc\in x\beta\subset x\beta^2$ and $yc\in xc\beta\subset x\beta^2$ whence $(xcb, ycb)\in U\times U\subset\alpha$ for all $b\in B$. Defining $A=cB$, it follows that $(xa, ya)\in\alpha$ for all $a\in A$. Since A is a syndetic subset of T, the proof is completed.

(2) Let α be an index such that $(xt, zt)\notin\alpha$ for all $z\in Z$ and all $t\in T$. Let β be a symmetric index of X such that $\beta^3\subset\alpha$. There exists a neighborhood U of x and a syndetic subset A of T such that $UA\subset x\beta$. If $y\in U$, if $z\in Z$, and if $a\in A$, then $(ya, za)\notin\beta$ since otherwise $(xa, za)=(xa, x)(x, ya)(ya, za)\in\beta^3\subset\alpha$ which is impossible. Let K be a compact subset of T such that $T=AK$. Choose an index γ of X such that $(x_1, x_2)\in\gamma$ implies $(x_1k^{-1}, x_2k^{-1})\in\beta$ for all $k\in K$. It follows that if $y\in U$ and $z\in Z$, then $(yt, zt)\notin\gamma$ for all $t\in T$. The proof is completed.

(3) By (2) there exists a neighborhood U of x such that U is distal from y. Again by (2) there exists a neighborhood V of y such that V is distal from U.

(4) By (3), $P'\subset Q'$ whence $P\supset Q$. Since also $P\subset Q$, we have $P=Q$. As remarked previously, P and Q are invariant reflexive symmetric, and Q is closed. It remains only to show that P is transitive.

Let $(x, y)\in P$ and $(y, z)\in P$. We show $(x, z)\in P$. Let α be an index of X. Choose an index β of X such that $\beta^2\subset\alpha$. By (1) there exists a syndetic subset A of T such that $(xa, ya)\in\beta$ for all $a\in A$. Let K be a compact subset of T such that $T=AK$. There exists an index γ of X such that $(x_1, x_2)\in\gamma$ implies $(x_1k^{-1}, x_2k^{-1})\in\beta$ for all $k\in K$. Choose $t\in T$ so that $(yt, zt)\in\gamma$. Write $t=bk$ where $b\in A$ and $k\in K$. Then $(yb, zb)=(ytk^{-1}, ztk^{-1})\in\beta$ and $(yb, zb)\in\beta$. Since also $(xb, yb)\in\beta$, it follows that $(xb, zb)=(xb, yb)(yb, zb)\in\beta^2\subset\alpha$ and $(xb, zb)\in\alpha$. Hence $(x, z)\in P$ and the proof is completed.

REMARK 13. Let T be locally almost periodic on X, and let $x, y\in X$ such that x is distal from y. Then xP is distal from yP. Moreover, some neighborhood of xP is distal from some neighborhood of yP.

COROLLARY 1. *If* T *is locally almost periodic on* X, *then* P *is an invariant closed equivalence relation in* X *and* T *is almost periodic on* $X|P$.

COROLLARY 2. *If* X *is a locally almost periodic minimal orbit-closure under*

T, then P is an invariant closed equivalence relation in X and $X|P$ is an almost periodic minimal orbit-closure under T.

THEOREM 3. *Let (X, T) be locally almost periodic. Then the following four relations in X all coincide:*

(1) *The proximal relation of (X, T).*
(2) *The regionally proximal relation of (X, T).*
(3) *The distal structure relation of (X, T).*
(4) *The equicontinuous structure relation of (X, T).*

Proof. Use (2) of Theorem 2 and (4) of Lemma 13.

DEFINITION 13. Let K be a circle in the plane. Then K^+ or K^- denotes the positively or negatively oriented circle K. The juxtaposition of K^+ or K^- and the customary notation for intervals $[a, b]$, $[a, b)$, $(a, b]$, (a, b) permits the designation of the closed, half-closed and half-open, open arcs on the circle K with endpoints a and b. Thus $K^-[a, b)$ is the arc of K extending from a to b in the negative direction and includes a but not b.

EXAMPLE 4 (Ellis). Let Y and Z be disjoint circles of radius 1 in the plane, let $X=Y\cup Z$, let τ be the permutation of X which translates Y onto Z and Z onto Y, and for each real number ϵ let λ_ϵ be the permutation of X which rotates each of Y and Z through ϵ radians. For $y\in Y$ and $0<\epsilon<\pi$ let $U(y, \epsilon) = Y^+[y, y\lambda_\epsilon) \cup Z^+(y\tau, y\tau\lambda_\epsilon]$. For $z\in Z$ and $0<\epsilon<\pi$ let $V(z, \epsilon) = Z^-[z, z\lambda_{-\epsilon})\cup Y^-(z\tau, z\tau\lambda_{-\epsilon}]$. Provide X with the topology such that $[U(y, \epsilon)|0<\epsilon<\pi]$ is an open-closed neighborhood base of each, $y\in Y$ and $[V(z, \epsilon)|0<\epsilon<\pi]$ is an open-closed neighborhood base of each $z\in Z$. The topological space X is a separable first-countable zero-dimensional compact Hausdorff space which is not second-countable and therefore not metrizable. Let $T=\mathcal{I}$ and define the transition π^1 to be λ_1. Clearly, π^1 is a homeomorphism of X onto X. Then (X, T) is a locally almost periodic minimal set such that $P=\Delta\cup[(x, x\tau)|x\in X]$ and indeed x is doubly asymptotic to $x\tau$ for every $x\in X$.

REMARK 14. Suppose (X, T, π) is equicontinuous. Then:

(1) The enveloping semigroup E of (X, T) is a compact topological group of homeomorphisms of X onto X.

(2) The original (point-index) topology of E coincides with the space-index topology of E; thus E equals the closure of the transition group G with respect to the space-index topology in the group of all homeomorphisms of X onto X.

(3) The map $\sigma: X\times E\rightarrow X$ defined by $(x, \xi)\sigma=x\xi(x\in X, \xi\in G)$ is continuous.

(4) (X, E, σ) is an equicontinuous transformation group.

(5) If $x\in X$, then $H_x=x\sigma_x^{-1}=[\xi|\xi\in E \text{ and } x\xi=x]$ is a closed subgroup of E where the motion $\sigma_x: E\rightarrow X$ is defined by $\xi\sigma_x=x\xi(\xi\in E)$.

Suppose further that (X, T) is point transitive, that is, there exists $x_0 \in X$ such that $[x_0 T]^- = X$. Then:

(6) (X, T) is minimal.

(7) If $x \in X$, then the motion σ_x maps E onto X. [Equicontinuity of (X, T) is not needed in (7)].

(8) If $x \in X$, then σ_x^{-1} defines an isomorphism of the transformation group (X, T) onto the transformation group $(E \backslash H_x, T)$ where $E \backslash H_x = [H_x \xi \mid \xi \in E]$ and $(H_x \xi)t = H_x(\xi \pi^t)(\xi \in E, t \in T)$.

Suppose further that T is abelian. Then:

(9) E is abelian. [Point transitivity of (X, T) is not needed in (9).]

(10) If $x \in X$, then H_x reduces to the identity map of X, σ_x is a homeomorphism of E onto X, and σ_x^{-1} is an isomorphism of the transformation group (X, T) onto the transformation group (E, T).

(11) If $x \in X$, then there exists a unique group structure in X which makes X a topological group and σ_x^{-1} a group isomorphism of X onto E.

(12) If $x \in X$, then there exists a unique group structure in X which makes X a topological group and π_x a group homomorphism of T into X.

Statements (1) and (2) follow from [1, Theorems 2 and 3]. The proof of the remaining statements is rather long altogether but offer no particular difficulty.

REMARK 15. If (X, T) is equicontinuous minimal, if $x \in X$, and if T is not abelian, then it may happen that $H_x = x\sigma_x^{-1}$ does not reduce to a single element. Let X be the 2-sphere, let T be the group of all isometries of X, and let T be provided with the space index topology. Then $S_e(X, T) = \Delta_x$ and $E = T$. However, H_x for $x \in X$ is not an invariant subgroup of E for if it were, then σ_x^{-1} would define a homeomorphism of X onto the topological group $E \backslash H_x$ which is impossible.

DEFINITION 14. The *structure group of* (X, T), denoted $\Gamma(X, T)$ or simply Γ, is defined to be the enveloping semigroup $E(X \mid S_e, T)$ of the equicontinuous structure transformation group $(X \mid S_e, T)$ of (X, T); here S_e denotes the equicontinuous structure relation of (X, T). Since $(X \mid S_e, T)$ is equicontinuous, the statements (1) to (5) of Remark 14 apply to $(X \mid S_e, T)$; if moreover (X, T) is point transitive, then $(X \mid S_e, T)$ is also point transitive and statements (6) to (8) apply to $(X \mid S_e, T)$; and if also T is abelian, then statements (9) to (12) apply to $(X \mid S_e, T)$.

REMARK 15. The notion of the structure group of a transformation group with compact phase space yields a partial classification of such transformation groups. In particular, a partial classification of minimal sets is provided. For example, the minimal set of Floyd [3] has the triadic group as its structure group. The following minimal sets have the circle group as their common structure group: the minimal set of Ellis [Example 4], the minimal set of Jones [4, 14.24], and the various Sturmian minimal sets [4, 12.63]. All of the above minimal sets are locally almost periodic discrete flows.

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